

q-deformed supersymmetric t - J model with a boundary

Bo-Yu Hou ^{a 1}, Wen-Li Yang ^{a,b 2}, Yao-Zhong Zhang ^{c 3} and Yi Zhen ^{a 4}

^a *Institute of Modern Physics, Northwest University, Xian 710069, China*

^b *Physikalisches Institut der Universitat Bonn, Nussallee 12, 53115 Bonn, Germany*

^c *Department of Mathematics, University of Queensland, Brisbane, Qld 4072, Australia*

Abstract

The q-deformed supersymmetric t - J model on a semi-infinite lattice is diagonalized by using the level-one vertex operators of the quantum affine superalgebra $U_q[\widehat{sl(2|1)}]$. We give the bosonization of the boundary states. We give an integral expression of the correlation functions of the boundary model, and derive the difference equations which they satisfy.

¹E-mail: byhou@phy.nwu.edu.cn

²E-mail: wlyang@phy.nwu.edu.cn

³E-mail: yzz@maths.uq.edu.au

⁴E-mail: zheny@phy.nwu.edu.cn

1 Introduction

Integrable models with quantum superalgebra symmetries have been the focus of recent studies [1, 2, 3, 4, 5, 6] in the context of strongly correlated fermion systems, a subject of high-profile international research activity because of their relevance to high- T_c superconductivity. The investigations to these models have largely been carried out within the framework of QISM and Bethe ansatz method. The exception are the works in [7], where the algebraic analysis method, developed in [8, 9] and generalized in [11, 12, 13, 14, 15], was used to diagonalize the supersymmetric t - J model and its multi-component version directly on an infinite lattice.

The algebraic analysis method [8, 9], which we will call the vertex operator method, was formulated with the help of the level-one q -vertex operators [10] and highest weight representations of quantum affine algebras. The vertex operator method was later extended in [16] to treat integrable models with boundary interactions [17, 18]. It was shown in [16] how the space of states of the boundary XXZ spin- $\frac{1}{2}$ chain on a semi-infinite lattice can be described in terms of level-one q -vertex operators of $U_q(\widehat{sl}_2)$, and how the correlation functions can be computed by the vertex operators. Several other models have been analysed by means of this approach [19, 20, 21, 22].

In this paper, we study the q -deformed supersymmetric t - J model with an integrable boundary. We will work directly on a semi-infinite lattice. As is known, the q -deformed supersymmetric t - J model on an infinite lattice (i.e. without a boundary) has as its symmetry algebra the quantum affine superalgebra $U_q[\widehat{sl}(2|1)]$ [7]. On a finite lattice with diagonal boundary reflection K -matrices this model was solved in [6] by the Bethe ansatz method. Here we adapt the vertex operator method. We will diagonalize the boundary model Hamiltonian directly on the semi-infinite lattice, and moreover compute the correlation functions of the boundary model.

This paper is organized as follows. In section 2, we describe the vertex operator approach to the q -deformed supersymmetric t - J model on the semi-infinite lattice. In section 3, we study the bosonic realization of the boundary states associated with the level-one highest weight representation of $U_q[\widehat{sl}(2|1)]$. In section 4, we compute the correlation functions of the local operators (including the spin operator S_1^z) and derive the difference equations which they satisfy. In appendix A, we review the bosonization of $U_q[\widehat{sl}(2|1)]$ at level-one and the associated vertex operators.

2 Boundary q -deformed supersymmetric t - J model

2.1 q -deformed supersymmetric t - J model on a finite lattice

In this section, we recall some facts about the q -deformed supersymmetric t - J model on a finite lattice. Throughout this paper, we fix q such that $|q| < 1$.

Let V be the 3-dimensional graded vector space and E_{ij} be the 3×3 matrix whose (i, j) -element is unity and zero otherwise. The grading of the basis vectors v_1, v_2, v_3 of V is chosen to be $[v_1] = [v_2] = 1, [v_3] = 0$. Let V^* be the dual space and $\{v_1^*, v_0^*, v_{-1}^*\}$ the dual basis vectors. Denote by V_z (resp. V_z^{*S}) the 3-dimensional level-0 representation (resp. dual representation) of $U_q[\widehat{sl}(2|1)]$ associated with V . Let $R(z) \in \text{End}(V \otimes V)$ be the R -matrix of $U_q[\widehat{sl}(2|1)]$ with matrix elements defined by

$$R(z)(v_i \otimes v_j) = \sum_{k,l} R_{kl}^{ij}(z) v_k \otimes v_l, \quad \forall v_i, v_j, v_k, v_l \in V,$$

where

$$\begin{aligned}
R_{33}^{33}\left(\frac{z_1}{z_2}\right) &= -\frac{z_1 q^{-1} - z_2 q}{z_1 q - z_2 q^{-1}}, \quad R_{23}^{23}\left(\frac{z_1}{z_2}\right) = -\frac{z_1 - z_2}{z_1 q - z_2 q^{-1}}, \quad R_{23}^{32}\left(\frac{z_1}{z_2}\right) = \frac{(q - q^{-1})z_2}{z_1 q - z_2 q^{-1}}, \\
R_{32}^{32}\left(\frac{z_1}{z_2}\right) &= -\frac{z_1 - z_2}{z_1 q - z_2 q^{-1}}, \quad R_{32}^{23}\left(\frac{z_1}{z_2}\right) = \frac{(q - q^{-1})z_1}{z_1 q - z_2 q^{-1}}, \quad R_{22}^{22}\left(\frac{z_1}{z_2}\right) = -1, \\
R_{13}^{13}\left(\frac{z_1}{z_2}\right) &= -\frac{z_1 - z_2}{z_1 q - z_2 q^{-1}}, \quad R_{13}^{31}\left(\frac{z_1}{z_2}\right) = \frac{(q - q^{-1})z_2}{z_1 q - z_2 q^{-1}}, \quad R_{31}^{31}\left(\frac{z_1}{z_2}\right) = -\frac{z_1 - z_2}{z_1 q - z_2 q^{-1}}, \\
R_{31}^{13}\left(\frac{z_1}{z_2}\right) &= \frac{(q - q^{-1})z_1}{z_1 q - z_2 q^{-1}}, \quad R_{12}^{12}\left(\frac{z_1}{z_2}\right) = -\frac{z_1 - z_2}{z_1 q - z_2 q^{-1}}, \quad R_{12}^{21}\left(\frac{z_1}{z_2}\right) = -\frac{(q - q^{-1})z_2}{z_1 q - z_2 q^{-1}}, \\
R_{21}^{21}\left(\frac{z_1}{z_2}\right) &= -\frac{z_1 - z_2}{z_1 q - z_2 q^{-1}}, \quad R_{21}^{12}\left(\frac{z_1}{z_2}\right) = -\frac{(q - q^{-1})z_1}{z_1 q - z_2 q^{-1}}, \quad R_{11}^{11}\left(\frac{z_1}{z_2}\right) = -1, \\
R_{kl}^{ij} &= 0, \quad \text{otherwise.}
\end{aligned}$$

The R-matrix satisfies the graded Yang-Baxter equation(YBE) on $V \otimes V \otimes V$

$$R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(z),$$

and moreover enjoys: (i) initial condition, $R(1) = P$ with P being the graded permutation operator; (ii) unitarity condition, $R_{12}(\frac{z}{w})R_{21}(\frac{w}{z}) = 1$, where $R_{21}(z) = PR_{12}(z)P$; and (iii) crossing-unitarity,

$$R^{-1, st_1}(z) \left((M \otimes 1)R(zq^{-2})(M \otimes 1) \right)^{st_1} = 1 \otimes 1,$$

where

$$M \equiv q^{2\bar{p}} \stackrel{def}{=} \begin{pmatrix} q^{2\rho_1} & & \\ & q^{2\rho_2} & \\ & & q^{2\rho_3} \end{pmatrix} = \begin{pmatrix} 1 & & \\ & q^{-2} & \\ & & q^{-2} \end{pmatrix}. \quad (2.1)$$

The various supertranspositions of the R-matrix are given by

$$\begin{aligned}
(R^{st_1}(z))_{ij}^{kl} &= R(z)_{kj}^{il}(-1)^{[i]([i]+[k])}, \quad (R^{st_2}(z))_{ij}^{kl} = R(z)_{il}^{kj}(-1)^{[j]([j]+[l])}, \\
(R^{st_{12}}(z))_{ij}^{kl} &= R(z)_{kl}^{ij}(-1)^{([i]+[j])([i]+[j]+[l]+[k])} = R(z)_{kl}^{ij}.
\end{aligned}$$

Following Sklyanin [17], we construct the transfer matrix of an integrable finite chain, with an open boundary condition described by a reflection K-matrix $K(z)$. Here $K(z)$ is a solution of the graded reflection equation

$$K_2(z_2)R_{21}(z_1 z_2)K_1(z_1)R_{12}(z_1/z_2) = R_{21}(z_1/z_2)K_1(z_1)R_{12}(z_1 z_2)K_2(z_2). \quad (2.2)$$

With appropriate normalization, we can show that this $K(z)$ obeys the relations

$$\begin{aligned}
K(1) &= 1, & (\text{Boundary initial condition}), \\
K(z)K(z^{-1}) &= 1, & (\text{Boundary unitarity}), \\
\overline{K}(z)\overline{K}(z^{-1}) &= 1, & (\text{Boundary crossing - unitarity}),
\end{aligned} \quad (2.3)$$

where $\overline{K}(z)$ is defined by

$$\overline{K}(z) = - \sum_{\alpha, \beta} R(z^2)_{i\beta}^{\alpha j}(-1)^{[i]+[j]+[j][\beta]+[\alpha][\beta]} K_{\alpha}^{\beta}(z^{-1}q^{-1})q^{2\rho_{\alpha}}. \quad (2.4)$$

The third relation is the graded extension of the boundary crossing-unitarity [18, 16, 23].

The transfer matrix of the q-deformed supersymmetric t - J model on a finite chain with the open boundary condition is constructed from $R(z)$ and $K(z)$ via [17, 24]

$$T_B^{\text{fin}}(z) = \text{str}_{V_0}(K^+(z)\mathcal{T}(z^{-1})K(z)\mathcal{T}(z)), \quad (2.5)$$

where $K^+(z) = K(-z^{-1}q^{-3})^{st}M$ and

$$\mathcal{T}(z) = R_{01}(z) \cdots R_{0N}(z) \in \text{End}(V_0 \otimes V_1 \otimes \cdots \otimes V_N)$$

is the double-row monodromy matrix. The supertrace is defined as $\text{str}(A) = \sum (-1)^{[i]} A_{ii}$.

It can be verified that $T_B^{\text{fin}}(z)$ form a commuting family, $[T_B^{\text{fin}}(z), T_B^{\text{fin}}(w)] = 0$. The Hamiltonian of the boundary q-deformed supersymmetric t - J model is given by [17, 6]

$$H_B^{\text{fin}} = \frac{d}{dz} T_B^{\text{fin}}(z)|_{z=1} = \sum_{j=1}^{N-1} h_{j,j+1} + \frac{1}{2} \frac{d}{dz} K(z)|_{z=1} + \frac{\text{str}_{V_0}(K^+(1)h_{0,N})}{K^+(1)}, \quad (2.6)$$

where $h_{j,j+1} = P_{j,j+1} \frac{d}{dz} R_{j,j+1}(z)|_{z=1}$.

The transfer matrix (2.5) with diagonal reflection K-matrices was diagonalized by the Bethe ansatz method in [6].

2.2 q-deformed supersymmetric t - J model on a semi-infinite lattice

In this paper, we restrict ourselves to the diagonal reflection K-matrix of the form

$$K(z) = f(z) \begin{pmatrix} \frac{1-rz}{z-r} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad f(z) = \frac{\phi(z|r)}{\phi(z^{-1}|r)}, \quad \phi(z|r) = \frac{1}{1-rz}, \quad (2.7)$$

where r is an arbitrary parameter which is related with the boundary interaction [18, 16]. One can check that such a K-matrix satisfies the boundary unitarity and crossing-unitarity (2.3).

We now consider Hamiltonian (2.6) in the semi-infinite limit:

$$H_B^{\text{fin}}|_{N \rightarrow \infty} = \sum_{j=1}^{\infty} h_{j,j+1} + \Delta, \quad (2.8)$$

where $\Delta = \frac{1}{2} \frac{d}{dz} f(z)|_{z=1}$ acts formally on the left-infinite tensor product space

$$\cdots \otimes V \otimes V. \quad (2.9)$$

As mentioned in the introduction, the q-deformed supersymmetric t - J model on an infinite lattice has $U_q[\widehat{sl(2|1)}]$ as its symmetry algebra. Let $V(\mu_\alpha)$ be the level-one irreducible highest weight $U_q[\widehat{sl(2|1)}]$ -modules with highest weight μ_α , $\alpha \in \mathbf{Z}$ (see (A.4) and [7]). Consider the level-one vertex operators which are intertwining operators between $V(\mu_\alpha)$ and $V(\mu_\beta)$. It has been shown in [7] that the following type I vertex operators $\Phi(z)$ exist, $\Phi^*(z)$ which interwine the level-one irreducible highest weight $U_q[\widehat{sl(2|1)}]$ -modules $V(\mu_\alpha)$

$$\Phi(z) : V(\mu_\alpha) \longrightarrow V(\mu_{\alpha-1}) \otimes V_z, \quad \Phi^*(z) : V(\mu_\alpha) \longrightarrow V(\mu_{\alpha+1}) \otimes V_z^{*S}. \quad (2.10)$$

(See Appendix A for more details about $V(\mu_\alpha)$ and its associated vertex operators.) Therefore following [8, 7, 16, 21], we can write the transfer matrix of the q -deformed supersymmetric t - J model on the semi-infinite lattice as

$$\begin{aligned} T_B(z) &= - \sum_{i,j=1}^3 \Phi_i^*(z^{-1}) K_i^j(z) \Phi_j(z) (-1)^{[i]} \\ &= \sum_{i,j=1}^3 q^{-2\rho_j} \Phi_j(z) \bar{K}_i^j(z^{-1} q^{-1}) \Phi_i^*(z^{-1}), \end{aligned} \quad (2.11)$$

where $\Phi_i(z)$ and $\Phi_j^*(z)$ are the components of the $U_q[\widehat{sl}(2|1)]$ vertex operators of type I (see (A.6)). We have used the exchange relations of vertex operators (A.14) and the definition of $\bar{K}(z)$ (2.4) in the above equation.

We remark that the transfer matrix $T_B(z)$ given by (2.11) is an operator with the property

$$T(z) : V(\mu_\alpha) \longrightarrow V(\mu_\alpha), \quad \alpha \in Z.$$

The commutativity of the transfer matrix (2.11), $[T_B(z), T_B(w)] = 0$, then follows from (A.12) and (2.2). Moreover by (A.12), (A.15) and (A.16), one can show

$$T_B(1) = id, \quad T_B(z) T_B(z^{-1}) = id, \quad (2.12)$$

$$T_B(z) T_B(z^{-1} q^{-2}) = id. \quad (2.13)$$

These relations correspond to the boundary initial condition, boundary unitarity and boundary crossing-unitarity (2.3) of the K -matrix, respectively. In terms of the transfer matrix, the q -deformed supersymmetric t - J model Hamiltonian on the semi-infinite lattice is given by

$$H = \frac{d}{dz} T_B(z) |_{z=1}. \quad (2.14)$$

Following [7], we define the local operators acting on the n -th site:

$$E_{i,j}^{(1)} = -\Phi_i^*(1) \Phi_j(1) (-1)^{[j]}, \quad (2.15)$$

$$E_{i,j}^{(n)} = \sum_m (-1)^{([i]+[j])[m]+[m]} \Phi_m^*(1) E_{i,j}^{(n-1)} \Phi_m(1), \quad n = 2, 3, \dots \quad (2.16)$$

In particular, we have the spin operator S_1^z

$$S_1^z = \frac{1}{2} (E_{11}^{(1)} - E_{22}^{(1)}) = \frac{1}{2} \{ \Phi_1^*(1) \Phi_1(1) - \Phi_2^*(1) \Phi_2(1) \}.$$

3 The boundary states

In this section we construct the bosonic boundary state $|\alpha; r \rangle_B$ and its dual state ${}_B \langle r; \alpha|$, which satisfy

$$T_B(z) |\alpha; r \rangle_B = |\alpha; r \rangle_B, \quad {}_B \langle r; \alpha| T_B(z) = {}_B \langle r; \alpha|. \quad (3.17)$$

By (A.15) and (2.11), the above eigenvalue problem is equivalent to

$$\Phi_i(z^{-1}) |\alpha; r \rangle_B = \sum_j K_i^j(z) \Phi_j(z) |\alpha; r \rangle_B, \quad (3.18)$$

$${}_B \langle r; \alpha| \Phi_j^*(z) (-1)^{[j]} = \sum_i {}_B \langle r; \alpha| \Phi_i^*(z^{-1}) K_i^j(z) (-1)^{[i]}. \quad (3.19)$$

3.1 The boundary state in $V(\Lambda_0)$

Firstly, we consider the boundary state $|0\rangle_B \in V(\mu_0)$ (or $V(\Lambda_0)$). As is shown in Appendix A, $V(\mu_0) = \eta_0 \xi_0 F_{(0;\beta)}$ and the highest weight vector $|\Lambda_0\rangle = |\beta, \beta, \beta, 0\rangle$ satisfies

$$\eta_0 |\Lambda_0\rangle = 0.$$

So we make the following ansatz [16]:

$$|0; r\rangle_B = e^{F_0(r)} |\Lambda_0\rangle, \quad (3.20)$$

$$F_0(r) = \frac{1}{2} \sum_{m=1}^{\infty} \frac{m}{[m]^2} \alpha_m \{h_{-m}^1 h_{-m}^{*1} + h_{-m}^2 h_{-m}^{*2} + c_{-m} c_{-m}\} + \sum_{m=1}^{\infty} \{\beta_m^1 h_{-m}^1 + \beta_m^2 h_{-m}^2 + \beta_m^3 c_{-m}\}, \quad (3.21)$$

where $\alpha_m, \beta_m^1, \beta_m^2, \beta_m^3$ are functions of the boundary parameter r .

We can check that $e^{F_0(r)}$ plays a role of the Bogoliubov transformation

$$\begin{aligned} e^{-F_0(r)} h_m^{*1} e^{F_0(r)} &= h_m^{*1} + \alpha_m h_{-m}^{*1} + \frac{[m]^2}{m} \beta_m^1, \\ e^{-F_0(r)} h_m^{*2} e^{F_0(r)} &= h_m^{*2} + \alpha_m h_{-m}^{*2} + \frac{[m]^2}{m} \beta_m^2, \\ e^{-F_0(r)} c_m e^{F_0(r)} &= c_m + \alpha_m c_{-m} + \frac{[m]^2}{m} \beta_m^3, \\ e^{-F_0(r)} h_m^1 e^{F_0(r)} &= h_m^1 + \alpha_m h_{-m}^1 + \frac{[2m][m]}{m} \beta_m^1 - \beta_m^2 \frac{[m]^2}{m}. \end{aligned}$$

Keeping (3.18) in mind and following [16, 19, 21], we find that the coefficients $\alpha_m, \beta_m^1, \beta_m^2, \beta_m^3$ are

$$\alpha_m = -q^{4m}, \quad \beta_m^1 = 0, \quad (3.22)$$

$$\beta_m^2 = \frac{r^m}{[m]} q^{\frac{3}{2}m} + \theta_m \frac{q^{\frac{5}{2}m} - q^{\frac{3}{2}m}}{[m]}, \quad (3.23)$$

$$\beta_m^3 = \theta_m \frac{q^{2m}}{[m]}, \quad (3.24)$$

where the function θ_m is defined by

$$\theta_m = \begin{cases} 1 & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases}.$$

Moreover following [19] one can check that $\eta_0 |0\rangle_B = 0$, namely, the boundary state $|0\rangle_B \in V(\mu_0)$, as required. In the derivation, the following relation are useful

$$\begin{aligned} e^{h_1^{*+}(\xi^{-1}q^2; -\frac{1}{2})} |0; r\rangle_B &= e^{h_1^{*-}(\xi q^2; -\frac{1}{2})} |0; r\rangle_B, \\ e^{-h_1^{+}(\omega q^2; -\frac{1}{2})} |0; r\rangle_B &= (1 - \omega^{-2})(1 - r\omega^{-1}) e^{-h_1^{-}(\omega^{-1}q^2; -\frac{1}{2})} |0; r\rangle_B, \\ e^{c^{+}(\omega q^2; 0)} |0\rangle_B &= (1 - \omega^{-2}) e^{c^{-}(\omega^{-1}q^2; 0)} |0; r\rangle_B, \\ e^{-h_2^{*+}(\xi^{-1}q^2; -\frac{1}{2})} |0\rangle_B &= (1 - \omega r)^{-1} e^{-h_2^{*-}(\xi q^2; -\frac{1}{2})} |0; r\rangle_B. \end{aligned}$$

Similarly, the dual state ${}_B \langle r; 0| \in V^*(\mu_0)$ can be constructed

$${}_B \langle r; 0| = \langle 0| e^{G_0(r)}, \quad (3.25)$$

$$\begin{aligned} G_0(r) = & -\frac{1}{2} \sum_{m=1}^{\infty} q^{-2m} \frac{m}{[m]^2} \{h_m^1 h_m^{1*} + h_m^2 h_m^{2*} + c_m c_m\} \\ & + \sum_{m=1}^{\infty} \{ \delta_m^1 h_m^1 + \delta_m^2 h_m^2 + \delta_m^3 c_m \}, \end{aligned} \quad (3.26)$$

where

$$\begin{aligned} \delta_m^1 &= 0, \\ \delta_m^2 &= -\frac{r^{-m} q^{-\frac{m}{2}}}{[m]} + \theta_m \left(\frac{q^{-\frac{m}{2}} + q^{-\frac{3}{2}m}}{[m]} \right), \\ \delta_m^3 &= \theta_m \left(\frac{q^{-m}}{[m]} \right). \end{aligned}$$

3.2 The general boundary states

Noting that the boundary K-matrix $K(z)$ have the following properties

$$K(z)|_{z=r} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad (3.27)$$

we may define $|-1; r \rangle_B = \Phi_1(r^{-1})|0; r \rangle_B|_{r \rightarrow rq^{-2}}$. One can check that such $|-1; r \rangle_B$ satisfies (3.18) with $\alpha = -1$. Recursively, we can construct the general boundary state $|\alpha; r \rangle_B$ from $|0; r \rangle_B$ by the following recursive relations,

$$|\alpha; rq^2 \rangle_B = \Phi_1(r^{-1})|\alpha + 1; r \rangle_B, \quad |\alpha; r \rangle_B = q^{-2\rho_1} \Phi_1^*(r^{-1}q^2)|\alpha - 1; rq^2 \rangle_B. \quad (3.28)$$

We have used the second invertibility relation (A.16). Similarly, we can obtain the dual boundary states ${}_B \langle r; \alpha|$ from ${}_B \langle r; 0|$ by the recursive relations,

$${}_B \langle r; \alpha| \Phi_1^*(r) = {}_B \langle rq^2; \alpha - 1|, \quad {}_B \langle r; \alpha| = {}_B \langle rq^2; \alpha - 1| \Phi_1(rq^{-2})q^{-2\rho_1}. \quad (3.29)$$

4 Correlation functions

The aim of this section is to calculate the one-point functions $\langle E_{i,j}^{(1)} \rangle_\alpha$:

$$\langle E_{i,j}^{(1)} \rangle_\alpha = \frac{{}_B \langle r; \alpha| E_{i,j}^{(1)} |\alpha; r \rangle_B}{{}_B \langle r; \alpha| \alpha; r \rangle_B}.$$

The generalization to the calculation of multi-point functions is straightforward. Thanks to the recursive relations (3.28) and (3.29), it is sufficient to calculate $\langle E_{i,j}^{(1)} \rangle_0$. Thus in the following we restrict ourselves to the calculation of $\langle E_{i,j}^{(1)} \rangle_0$.

Define

$$\oint dz f(z) = f_{-1}, \quad \text{for formal series function } f(z) = \sum_{n \in \mathbb{Z}} f_n z^n.$$

By the bosonic realization of Drinfeld currents of $U_q[\widehat{sl(2|1)}]$, (A.7)-(A.10) and the normal ordering relations in the appendix A, we obtain the integral expression of the vertex operators [7]

$$\begin{aligned}
\phi_3(z) &= : e^{-h_2^*(q^2 z; -\frac{1}{2}) + c(q^2 z; 0)} : e^{-i\pi a_0^2}, \\
\phi_2(z) &= \left\{ \frac{e^{-c(wq; 0)}}{wq(1 - \frac{qz}{w})} + \frac{e^{-c(wq^{-1}; 0)}}{zq^2(1 - \frac{w}{zq^3})} \right\} e^{-h_2^*(q^2 z; -\frac{1}{2}) - h_2(w; -\frac{1}{2}) + c(q^2 z; 0)} e^{i\pi h_0^1}, \\
\phi_1(z) &= \frac{q^2 - 1}{w(1 - \frac{w_1 q}{w})(1 - \frac{wq}{w_1})} \\
&\quad \times : \left\{ \frac{e^{-c(wq; 0)}}{wq(1 - \frac{qz}{w})} + \frac{e^{-c(wq^{-1}; 0)}}{zq^2(1 - \frac{w}{zq^3})} \right\} e^{-h_2^*(q^2 z; -\frac{1}{2}) - h_2(w; -\frac{1}{2}) - h_1(w_1; -\frac{1}{2}) + c(q^2 z; 0)} : e^{-i\pi a_0^2}, \\
\phi_1^*(z) &= : e^{h_1^*(qz; -\frac{1}{2})} : e^{i\pi a_0^2}, \\
\phi_2^*(z) &= \oint dw \frac{1 - q^{-2}}{z(1 - \frac{zq^2}{w})(1 - \frac{w}{z})} : e^{h_1^*(qz; -\frac{1}{2}) - h_1(w; -\frac{1}{2})} e^{-i\pi h_0^1}, \\
\phi_3^*(z) &= \oint dw_1 \oint dw \frac{1 - q^{-2}}{z(1 - \frac{zq^2}{w})(1 - \frac{w}{z})} \\
&\quad \times : \frac{e^{-c(w_1 q; 0)} - e^{-c(w_1 q^{-1}; 0)}}{ww_1(1 - \frac{wq}{w_1})(1 - \frac{w_1 q}{w})} e^{h_1^*(qz; -\frac{1}{2}) - h_1(w; -\frac{1}{2}) - h_2(w_1; -\frac{1}{2})} e^{i\pi a_0^2} :
\end{aligned}$$

Since $\eta_0|0, r\rangle = 0$, one may set

$$P_{i,j}(z_1, z_2) = \frac{{}_B\langle r; 0|\Phi_i^*(z_1)\Phi_j(z_2)|0; r\rangle_B}{{}_B\langle r; 0|0; r\rangle_B} \equiv \frac{{}_B\langle r; 0|\phi_i^*(z_1)\phi_j(z_2)|0; r\rangle_B}{{}_B\langle r; 0|0; r\rangle_B}, \quad (4.1)$$

then $\langle E_{i,j}^{(1)} \rangle_0 = -(-1)^{[j]} P_{i,j}(1, 1)$.

The bosonization formulae (A.7)-(A.10) of the vertex operators immediately imply

$$P_{i,j}(z_1, z_2) = \delta_{ij} F_i(z_1, z_2) \stackrel{def}{=} \delta_{ij} \frac{{}_B\langle r; 0|\phi_i^*(z_1)\phi_i(z_2)|0; r\rangle_B}{{}_B\langle r; 0|0; r\rangle_B}.$$

Using the technique in [16, 21] (see equation (C.4)), after tedious calculation, we get

$$\begin{aligned}
{}_B\langle r; 0|0; r\rangle_B &= \prod_{n=1}^{\infty} \frac{1}{1 - \alpha_n \gamma_n} \prod_{n=1}^{\infty} \frac{1}{(\alpha_n \gamma_n - 1)^{\frac{1}{2}}} \\
&\quad \exp \left[\frac{1}{2} \sum \frac{[n]^2}{n} \frac{1}{1 - \alpha_n \gamma_n} (\gamma_n (\beta_n^3)^2 + 2\beta_n^3 \delta_n^3 + \alpha_n (\delta_n^3)^2) \right], \quad (4.2)
\end{aligned}$$

$$\begin{aligned}
F_1(z_1, z_2) &= \frac{1}{{}_B\langle r; 0|0; r\rangle_B} \oint d\omega_1 \oint d\omega \frac{(q^2 - 1)g_1}{q\omega^2(1 - \frac{\omega_1 q}{\omega})(1 - \frac{\omega q}{\omega_1})(1 - \frac{z_2 q}{\omega})} \\
&\quad \times \prod_{n=1}^{\infty} (-(\alpha_n \gamma_n - 1)^{-1}) \prod_{n=1}^{\infty} (\alpha_n \gamma_n - 1)^{-\frac{1}{2}} \\
&\quad \times \exp \left(\sum \frac{[n]^2}{n} \frac{1}{(\alpha_n \gamma_n - 1)} \left\{ (B_1 - C_1)^2 \frac{[2n]}{[n]} \frac{\gamma_n}{\alpha_n \gamma_n - 1} + \gamma_n (B_1 - C_1) \beta_n^2 \right. \right. \\
&\quad \left. \left. - \gamma_n (B_1 - C_1) A_1 + (B_1 - C_1) \delta_n^2 \right\} \right)
\end{aligned}$$

$$\begin{aligned}
& \times \exp\left(\sum \frac{[n]^2}{n} \frac{1}{1 - \alpha_n \gamma_n} \left\{ \frac{1}{2} (\beta_n^3)^2 D \gamma_n + \frac{1}{2} \beta_n^3 D_1^2 \gamma_n + \beta_n^3 D_1 \gamma_n + \beta_n^3 \delta_n^3 \right. \right. \\
& \quad \left. \left. + D_1 \delta_n^3 + \frac{1}{2} \alpha_n (\delta_n^3)^2 \right\} \right) \\
& + \oint d\omega_1 \oint d\omega \frac{(q^2 - 1) g'_1}{q^2 \omega z_2 (1 - \frac{\omega_1 q}{\omega}) (1 - \frac{\omega q}{\omega_1}) (1 - \frac{\omega}{z_2 q^3})} \\
& \times \prod_{n=1}^{\infty} (- (\alpha_n \gamma_n - 1)^{-1}) \prod_{n=1}^{\infty} (\alpha_n \gamma_n - 1)^{-\frac{1}{2}} \\
& \times \exp\left(\sum \frac{[n]^2}{n} \frac{1}{(\alpha_n \gamma_n - 1)} \left\{ (B_1 - C_1)^2 \frac{[2n]}{[n]} \frac{\gamma_n}{\alpha_n \gamma_n - 1} + \gamma_n (B_1 - C_1) \beta_n^2 \right. \right. \\
& \quad \left. \left. - \gamma_n (B_1 - C_1) A_1 + (B_1 - C_1) \delta_n^2 \right\} \right) \\
& \times \exp\left(\sum \frac{[n]^2}{n} \frac{1}{1 - \alpha_n \gamma_n} \left\{ \frac{1}{2} (\beta_n^3)^2 D'_1 \gamma_n + \frac{1}{2} \beta_n^3 (D'_1)^2 \gamma_n + \beta_n^3 D'_1 \gamma_n + \beta_n^3 \delta_n^3 \right. \right. \\
& \quad \left. \left. + D'_1 \delta_n^3 + \frac{1}{2} \alpha_n (\delta_n^3)^2 \right\} \right), \tag{4.3}
\end{aligned}$$

where

$$\begin{aligned}
g_1 = & \exp\left(-\sum \frac{q^{3n} z_2^{-n} \omega^{-n}}{n}\right) \exp\left(\sum \frac{q^n z_2^{-n} \omega^{-n}}{n}\right) \exp\left(\sum \frac{q^n z_2^n \omega^{-n}}{n}\right) \\
& \exp\left(-\sum \frac{q^{-n} z_2^{-n} \omega^n}{n}\right) \exp\left(\sum \frac{r^n z_2^{-n}}{n}\right) \exp\left(\sum \frac{q^{5n} \omega^{-n} \omega_1^{-n}}{n}\right) \\
& \exp\left(\sum \frac{q^{4n} z_1^{-n} \omega_1^{-n}}{n}\right) \exp\left(\sum \frac{q^n \omega^{-n} \omega_1^n}{n}\right) \exp\left(-\sum \frac{q^{4n} \omega_1^{-2n}}{n}\right) \\
& \exp\left(-\sum \frac{r^n q^{2n} \omega_1^{-n}}{n}\right) \exp\left(-\sum \frac{q^{2n} z_2^{-n} z_1^{-n}}{n}\right) \exp\left(-\sum \frac{q^{2n} z_2^n z_1^{-n}}{n}\right) \\
& \exp\left(\sum \frac{z_1^{-n} \omega_1^n}{n}\right),
\end{aligned}$$

$$\begin{aligned}
g'_1 = & \exp\left(\sum \frac{q^n z_2^n \omega^{-n}}{n}\right) \exp\left(-\sum \frac{q^{-n} z_2^{-n} \omega^n}{n}\right) \exp\left(\sum \frac{r^n z_2^{-n}}{n}\right) \\
& \times \exp\left(\sum \frac{q^{5n} \omega^{-n} \omega_1^{-n}}{n}\right) \exp\left(\sum \frac{q^{4n} z_1^{-n} \omega_1^{-n}}{n}\right) \exp\left(\sum \frac{q^n \omega^{-n} \omega_1^n}{n}\right) \\
& \times \exp\left(-\sum \frac{q^{4n} \omega_1^{-2n}}{n}\right) \exp\left(-\sum \frac{r^n q^{2n} \omega_1^{-n}}{n}\right) \exp\left(-\sum \frac{q^{2n} z_2^{-n} z_1^{-n}}{n}\right) \\
& \times \exp\left(-\sum \frac{q^{2n} z_2^n z_1^{-n}}{n}\right) \exp\left(\sum \frac{z_1^{-n} \omega_1^n}{n}\right),
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= \sum \frac{q^{\frac{3}{2}n} z_1^n}{[n]} - \alpha_n \sum \frac{q^{-\frac{1}{2}n} z_1^{-n}}{[n]} + \sum \frac{q^{\frac{1}{2}n} \omega^n}{[n]} - \alpha_n \sum \frac{q^{\frac{1}{2}n} \omega^{-n}}{[n]}, \\
B_1 &= \sum \frac{q^{\frac{5}{2}n} z_2^n}{[n]} - \alpha_n \sum \frac{q^{-\frac{3}{2}n} z_2^{-n}}{[n]}, \\
D_1 &= \sum \frac{q^{2n} z_2^n}{[n]} - \alpha_n \sum \frac{q^{-2n} z_2^{-n}}{[n]} - \sum \frac{q^n \omega^n}{[n]} + \alpha_n \sum \frac{q^{-n} \omega^{-n}}{[n]},
\end{aligned}$$

$$\begin{aligned}
C_1 &= \sum \frac{q^{\frac{1}{2}n} \omega_1^n}{[n]} - \alpha_n \sum \frac{q^{\frac{1}{2}n} \omega_1^{-n}}{[n]}, \\
D_1' &= \sum \frac{q^{2n} z_2^n}{[n]} - \alpha_n \sum \frac{q^{-2n} z_2^{-n}}{[n]} - \sum \frac{q^{-n} \omega^n}{[n]} + \alpha_n \sum \frac{q^n \omega^{-n}}{[n]},
\end{aligned}$$

$$\begin{aligned}
F_2(z_1, z_2) &= \frac{1}{B\langle r; 0|0; r \rangle_B} \oint d\omega \oint d\omega_1 \frac{(1 - q^{-2}) g_2}{z_1 \left(1 - \frac{z_1 q^2}{\omega}\right) \left(1 - \frac{\omega}{z_1}\right) \omega_1 q \left(1 - \frac{z_2 q}{\omega_1}\right)} \\
&\times \prod_{n=1}^{\infty} (-(\alpha_n \gamma_n - 1)^{-1}) \prod_{n=1}^{\infty} (\alpha_n \gamma_n - 1)^{-\frac{1}{2}} \\
&\times \exp \left\{ \sum \frac{[n]^2}{n} \frac{1}{\alpha_n \gamma_n - 1} [(B_2 - C_2)^2 \frac{[2n]}{[n]} \frac{\gamma_n}{\alpha_n \gamma_n - 1} + \gamma_n (B_2 - C_2) \beta_n^2 \right. \\
&\quad \left. - \gamma_n (B_2 - C_2) A_2 + (B_2 - C_2) \delta_n^2] \right\} \\
&\times \exp \left\{ \sum \frac{[n]^2}{n} \frac{1}{1 - \alpha_n \gamma_n} \left[\frac{1}{2} (\beta_n^3)^2 D_2 \gamma_n + \frac{1}{2} \beta_n^3 D_2^2 \gamma_n + \beta_n^3 D_2 \gamma_n + \beta_n^3 \delta_n^3 \right. \right. \\
&\quad \left. \left. + D_2 \delta_n^3 + \frac{1}{2} \alpha_n (\delta_n^3)^2 \right] \right\} \\
&+ \oint d\omega \oint d\omega_1 \frac{(1 - q^{-2}) g_2'}{z_1 \left(1 - \frac{z_1 q^2}{\omega}\right) \left(1 - \frac{\omega}{z_1}\right) z_2 q^2 \left(1 - \frac{\omega_1}{z_2 q^3}\right)} \\
&\times \exp \left\{ \sum \frac{[n]^2}{n} \frac{1}{\alpha_n \gamma_n - 1} [(B_2 - C_2)^2 \frac{[2n]}{[n]} \frac{\gamma_n}{\alpha_n \gamma_n - 1} + \gamma_n (B_2 - C_2) \beta_n^2 \right. \\
&\quad \left. - \gamma_n (B_2 - C_2) A_2 + (B_2 - C_2) \delta_n^2] \right\} \\
&\times \prod_{n=1}^{\infty} (-(\alpha_n \gamma_n - 1)^{-1}) \prod_{n=1}^{\infty} (\alpha_n \gamma_n - 1)^{-\frac{1}{2}} \\
&\times \exp \left\{ \sum \frac{[n]^2}{n} \frac{1}{1 - \alpha_n \gamma_n} \left[\frac{1}{2} (\beta_n^3)^2 D_2' \gamma_n + \frac{1}{2} \beta_n^3 D_2'^2 \gamma_n + \beta_n^3 D_2' \gamma_n + \beta_n^3 \delta_n^3 \right. \right. \\
&\quad \left. \left. + D_2' \delta_n^3 + \frac{1}{2} \alpha_n (\delta_n^3)^2 \right] \right\}, \tag{4.4}
\end{aligned}$$

where

$$\begin{aligned}
g_2 &= \exp \left(\sum \frac{\omega^n z_1^{-n}}{n} \right) \exp \left(- \sum \frac{q^{2n} z_1^{-n} z_2^n}{n} \right) \exp \left(\sum \frac{q^n \omega^{-n} \omega_1^n}{n} \right) \\
&\times \exp \left(- \sum \frac{q^{-n} \omega_1^n z_2^{-n}}{n} \right) \exp \left(\sum \frac{q^n \omega_1^{-n} z_2^n}{n} \right) \exp \left(\sum \frac{r^n z_2^{-n}}{n} \right) \\
&\times \exp \left(- \sum \frac{\omega^{-2n} q^{4n}}{n} \right) \exp \left(- \sum \frac{\omega^{-n} q^{2n} r^n}{n} \right) \exp \left(\sum \frac{q^{4n} \omega^{-n} z_1^{-n}}{n} \right) \\
&\times \exp \left(\sum \frac{q^{5n} \omega^{-n} \omega_1^{-n}}{n} \right) \exp \left(- \sum \frac{q^{3n} \omega_1^{-n} z_2^{-n}}{n} \right) \exp \left(\sum \frac{q^n \omega_1^{-n} z_2^{-n}}{n} \right) \\
&\times \exp \left(- \sum \frac{q^{2n} z_2^{-n} z_1^{-n}}{n} \right),
\end{aligned}$$

$$g_2' = \exp \left(\sum \frac{\omega^n z_1^{-n}}{n} \right) \exp \left(- \sum \frac{q^{2n} z_2^n z_1^{-n}}{n} \right) \exp \left(\sum \frac{q^n \omega^{-n} \omega_1^n}{n} \right)$$

$$\begin{aligned}
& \times \exp\left(-\sum \frac{q^{-n}\omega_1^n z_2^{-n}}{n}\right) \exp\left(\sum \frac{q^{3n}\omega_1^{-n} z_2^n}{n}\right) \exp\left(\sum \frac{r^n z_2^{-n}}{n}\right) \\
& \times \exp\left(-\sum \frac{\omega^{-2n} q^{4n}}{n}\right) \exp\left(-\sum \frac{\omega^{-n} q^{2n} r^n}{n}\right) \exp\left(\sum \frac{q^{4n} \omega^{-n} z_1^{-n}}{n}\right) \\
& \times \exp\left(\sum \frac{q^{5n} \omega^{-n} \omega_1^{-n}}{n}\right) \exp\left(-\sum \frac{q^{2n} z_2^{-n} z_1^{-n}}{n}\right),
\end{aligned}$$

and

$$\begin{aligned}
A_2 &= \sum \frac{q^{\frac{3}{2}n} z_1^n}{[n]} - \alpha_n \sum \frac{q^{-\frac{1}{2}n} z_1^{-n}}{[n]} + \sum \frac{q^{\frac{1}{2}n} \omega_1^n}{[n]} - \alpha_n \sum \frac{q^{\frac{1}{2}n} \omega_1^{-n}}{[n]}, \\
B_2 &= \sum \frac{q^{\frac{5}{2}n} z_2^n}{[n]} - \alpha_n \sum \frac{q^{-\frac{3}{2}n} z_2^{-n}}{[n]}, \\
D_2 &= \sum \frac{q^{2n} z_2^n}{[n]} - \alpha_n \sum \frac{q^{-2n} z_2^{-n}}{[n]} - \sum \frac{q^n \omega_1^n}{[n]} + \alpha_n \sum \frac{q^{-n} \omega_1^{-n}}{[n]}, \\
C_2 &= \sum \frac{q^{\frac{1}{2}n} \omega_1^n}{[n]} - \alpha_n \sum \frac{q^{\frac{1}{2}n} \omega_1^{-n}}{[n]}, \\
D_2' &= \sum \frac{q^{2n} z_2^n}{[n]} - \alpha_n \sum \frac{q^{-2n} z_2^{-n}}{[n]} - \sum \frac{q^{-n} \omega_1^n}{[n]} + \alpha_n \sum \frac{q^n \omega_1^{-n}}{[n]},
\end{aligned}$$

$$\begin{aligned}
F_3(z_1, z_2) &= \frac{-1}{B\langle r; 0|0; r\rangle_B} \oint d\omega \oint d\omega_1 \frac{(1-q^{-2})g_3}{z_1 \left(1 - \frac{z_1 q^2}{\omega}\right) \left(1 - \frac{\omega}{z_1}\right) \omega_1 \omega \left(1 - \frac{\omega q}{\omega_1}\right) \left(1 - \frac{\omega_1 q}{\omega}\right)} \\
&\times \prod_{n=1}^{\infty} (-\alpha_n \gamma_n - 1)^{-1} \prod_{n=1}^{\infty} (\alpha_n \gamma_n - 1)^{-\frac{1}{2}} \\
&\times \exp\left\{\sum \frac{[n]^2}{n} \frac{1}{\alpha_n \gamma_n - 1} [(B_3 - C_3)^2 \frac{[2n]}{[n]} \frac{\gamma_n}{\alpha_n \gamma_n - 1} + \gamma_n (B_3 - C_3) \beta_n^2 \right. \\
&\quad \left. - \gamma_n (B_3 - C_3) A_3 + (B_3 - C_3) \delta_n^2]\right\} \\
&\times \exp\left\{\sum \frac{[n]^2}{n} \frac{1}{1 - \alpha_n \gamma_n} \left[\frac{1}{2} (\beta_n^3)^2 D_3 \gamma_n + \frac{1}{2} \beta_n^3 D_3^2 \gamma_n + \beta_n^3 D_3 \gamma_n + \beta_n^3 \delta_n^3 \right. \right. \\
&\quad \left. \left. + D_3 \delta_n^3 + \frac{1}{2} \alpha_n (\delta_n^3)^2\right]\right\} \\
&+ \oint d\omega \oint d\omega_1 \frac{(1-q^{-2})g_3'}{z_1 \left(1 - \frac{z_1 q^2}{\omega}\right) \left(1 - \frac{\omega}{z_1}\right) \omega \omega_1 \left(1 - \frac{\omega q}{\omega_1}\right) \left(1 - \frac{\omega_1 q}{\omega}\right)} \\
&\times \prod_{n=1}^{\infty} (-\alpha_n \gamma_n - 1)^{-1} \prod_{n=1}^{\infty} (\alpha_n \gamma_n - 1)^{-\frac{1}{2}} \\
&\times \exp\left\{\sum \frac{[n]^2}{n} \frac{1}{\alpha_n \gamma_n - 1} [(B_3 - C_3)^2 \frac{[2n]}{[n]} \frac{\gamma_n}{\alpha_n \gamma_n - 1} + \gamma_n (B_3 - C_3) \beta_n^2 \right. \\
&\quad \left. - \gamma_n (B_3 - C_3) A_3 + (B_3 - C_3) \delta_n^2]\right\} \\
&\times \exp\left\{\sum \frac{[n]^2}{n} \frac{1}{1 - \alpha_n \gamma_n} \left[\frac{1}{2} (\beta_n^3)^2 D_3' \gamma_n + \frac{1}{2} \beta_n^3 D_3'^2 \gamma_n + \beta_n^3 D_3' \gamma_n + \beta_n^3 \delta_n^3 \right. \right. \\
&\quad \left. \left. + D_3' \delta_n^3 + \frac{1}{2} \alpha_n (\delta_n^3)^2\right]\right\}, \tag{4.5}
\end{aligned}$$

where

$$g_3 = \exp\left(-\sum \frac{\omega^n z_1^{-n}}{n}\right) \left(-\sum \frac{q^{2n} z_1^{-n} z_2^n}{n}\right) \exp\left(\sum \frac{q^n \omega^{-n} \omega_1^n}{n}\right)$$

$$\begin{aligned}
& \times \exp\left(-\sum \frac{q^{3n}\omega_1^{-n}z_2^n}{n}\right) \exp\left(\sum \frac{q^n\omega_1^{-n}z_2^n}{n}\right) \exp\left(\sum \frac{r^n z_2^{-n}}{n}\right) \\
& \times \exp\left(-\sum \frac{\omega^{-2n}q^{4n}}{n}\right) \exp\left(-\sum \frac{\omega^{-n}q^{2n}r^n}{n}\right) \exp\left(\sum \frac{q^{4n}\omega^{-n}z_1^{-n}}{n}\right) \\
& \times \exp\left(\sum \frac{q^{5n}\omega^{-n}\omega_1^{-n}}{n}\right) \exp\left(-\sum \frac{q^{3n}\omega_1^{-n}z_2^{-n}}{n}\right) \exp\left(\sum \frac{q^n\omega_1^{-n}z_2^{-n}}{n}\right) \\
& \times \exp\left(-\sum \frac{q^{2n}z_2^{-n}z_1^{-n}}{n}\right), \\
g'_3 = & \exp\left(-\sum \frac{\omega^n z_1^{-n}}{n}\right) \exp\left(\sum \frac{q^n\omega^{-n}\omega_1^n}{n}\right) \left(-\sum \frac{q^{2n}z_1^{-n}z_2^n}{n}\right) \\
& \times \exp\left(\sum \frac{r^n z_2^{-n}}{n}\right) \exp\left(-\sum \frac{\omega^{-2n}q^{4n}}{n}\right) \exp\left(-\sum \frac{\omega^{-n}q^{2n}r^n}{n}\right) \\
& \times \exp\left(-\sum \frac{q^{2n}z_2^{-n}z_1^{-n}}{n}\right) \exp\left(\sum \frac{q^{5n}\omega^{-n}\omega_1^{-n}}{n}\right) \exp\left(\sum \frac{q^{4n}\omega^{-n}z_1^{-n}}{n}\right),
\end{aligned}$$

and

$$\begin{aligned}
A_3 &= \sum \frac{q^{\frac{3}{2}n}z_1^n}{[n]} - \alpha_n \sum \frac{q^{-\frac{1}{2}n}z_1^{-n}}{[n]} + \sum \frac{q^{\frac{1}{2}n}\omega_1^n}{[n]} - \alpha_n \sum \frac{q^{\frac{1}{2}n}\omega_1^{-n}}{[n]}, \\
B_3 &= \sum \frac{q^{\frac{5}{2}n}z_2^n}{[n]} - \alpha_n \sum \frac{q^{-\frac{3}{2}n}z_2^{-n}}{[n]}, \\
D_3 &= \sum \frac{q^{2n}z_2^n}{[n]} - \alpha_n \sum \frac{q^{-2n}z_2^{-n}}{[n]} - \sum \frac{q^n\omega_1^n}{[n]} + \alpha_n \sum \frac{q^{-n}\omega_1^{-n}}{[n]}, \\
C_3 &= \sum \frac{q^{\frac{1}{2}n}\omega_1^n}{[n]} - \alpha_n \sum \frac{q^{\frac{1}{2}n}\omega_1^{-n}}{[n]}, \\
D'_3 &= \sum \frac{q^{2n}z_2^n}{[n]} - \alpha_n \sum \frac{q^{-2n}z_2^{-n}}{[n]} - \sum \frac{q^{-n}\omega_1^n}{[n]} + \alpha_n \sum \frac{q^n\omega_1^{-n}}{[n]}.
\end{aligned}$$

We now derive the difference equations satisfied by the one-point functions. By (2.11) and (A.15)-(A.16), one obtains

$$\Phi_i^*(z^{-1})|\alpha; r \rangle_B = \sum_j \bar{K}_i^j(zq)\Phi_j^*(zq^2)|\alpha; r \rangle_B, \quad (4.6)$$

$${}_B \langle r; \alpha | \Phi_i(z)(-1)^{[i]} = \sum_j {}_B \langle r; \alpha | \Phi_j(z^{-1}q^{-2})\bar{K}_i^j(zq)(-1)^{[j]}. \quad (4.7)$$

By (A.14), one derives the exchange relations

$$\Phi_i^*(z_1)\Phi_j(z_2) = \sum_{kl} \tilde{R} \left(\frac{z_1}{z_2} \right)_{ij}^{kl} \Phi_l(z_2)\Phi_k^*(z_1)(-1)^{[k][l]}, \quad (4.8)$$

where $\tilde{R}(z) = R^{-1, st_1, -1}(z)$.

Using (3.18)-(3.19), (4.6)-(4.8), (A.14) and (A.7)-(A.10), we get the difference equations

$$\begin{aligned}
F_i(z_1q^{-2}, z_2) &= \sum_{j,k,l,m,n} (-1)^{[k][l]+[i]+[j]+[n]} K_j^i(z_1q^{-2}) \tilde{R} \left(\frac{z_1}{z_2} \right)_{ji}^{lk} \\
&\quad \times \bar{K}_l^m(z_1q^{-1}) \bar{R} \left(\frac{z_1}{z_2} \right)_{m\ n}^{n\ k} F_n(z_1, z_2),
\end{aligned} \quad (4.9)$$

$$\begin{aligned}
F_i(z_1, z_2 q^2) &= \sum_{j,k,l,m,n} (-1)^{[k][l]+[l]+[m]+[n]} K_i^j(z_2^{-1} q^{-2}) \tilde{R}(z_1 z_2 q^2)_{kl}^{ij} \\
&\quad \times \bar{K}_l^m(z_2^{-1} q^{-1}) \bar{R}\left(\frac{z_1}{z_2}\right)_k^n \bar{F}_m^n(z_1, z_2). \quad (4.10)
\end{aligned}$$

Acknowledgment

We would like to thank G. von Gehlen for his interest in this problem. This work has been partly supported by the grant of National Natural Science Foundation of China. W.-L. Yang is supported by the Alexander von Humboldt Foundation. Y.-Z. Zhang is supported by Australian Research Council.

A Appendix A

A.1 Bosonization of $U_q[\widehat{sl(2|1)}]$

In this section, we briefly review the bosonization of $U_q[\widehat{sl(2|1)}]$ at level-one and the corresponding vertex operators [25, 7].

The Cartan matrix of $U_q[\widehat{sl(2|1)}]$ is

$$(a_{ij}) = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

where $i, j = 0, 1, 2$.

In terms of the Drinfeld generators: $\{d, X_m^{\pm, i}, h_n^i, (K^i)^{\pm 1}, \gamma^{\pm 1/2} | i = 1, 2, m \in \mathbf{Z}, n \in \mathbf{Z}_{\neq 0}\}$, the defining relations of $U_q[\widehat{sl(2|1)}]$ read

$$\begin{aligned}
&\gamma \text{ is central, } [K^i, h_m^j] = 0, [d, K^i] = 0, [d, h_m^j] = m h_m^j, \\
&[h_m^i, h_n^j] = \delta_{m+n, 0} \frac{[a_{ij} m](\gamma^m - \gamma^{-m})}{m(q - q^{-1})}, \\
&K^i X_m^{\pm, j} = q^{\pm a_{ij}} X_m^{\pm, j} K^i, [d, X_m^{\pm, j}] = m X_m^{\pm, j}, \\
&[h_m^i, X_n^{\pm, j}] = \pm \frac{[a_{ij} m]}{m} \gamma^{\pm |m|/2} X_{n+m}^{\pm, j}, \\
&[X_m^{+, i}, X_n^{-, j}] = \frac{\delta_{i,j}}{q - q^{-1}} (\gamma^{(m-n)/2} \psi_{m+n}^{+, j} - \gamma^{-(m-n)/2} \psi_{m+n}^{-, j}), \\
&[X_m^{\pm, 2}, X_n^{\pm, 2}] = 0, \\
&[X_{m+1}^{\pm, i}, X_n^{\pm, j}]_{q^{\pm a_{ij}}} + [X_{n+1}^{\pm, j}, X_m^{\pm, i}]_{q^{\pm a_{ij}}} = 0, \text{ for } a_{ij} \neq 0,
\end{aligned}$$

where $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$, $[X, Y]_{\xi} = XY - (-1)^{[X][Y]} \xi YX$ and $[X, Y]_1 \equiv [X, Y]$; the \mathbf{Z}_2 -grading of Drinfeld generators are : $[X_m^{\pm, 2}] = 1$ for $m \in \mathbf{Z}$ and zero otherwise.

Introduce the bosonic q-oscillators [25] $\{a_n^1, a_n^2, b_n, c_n, Q_{a^1}, Q_{a^2}, Q_b, Q_c \mid n \in \mathbf{Z}\}$, which satisfy the commutation relations

$$[a_m^i, a_n^j] = \delta_{i,j} \delta_{m+n, 0} \frac{[m]^2}{m}, \quad [a_0^i, Q_{a^j}] = \delta_{i,j},$$

$$[b_m, b_n] = -\delta_{m+n,0} \frac{[m]^2}{m}, \quad [b_0, Q_b] = -1,$$

$$[c_m, c_n] = \delta_{m+n,0} \frac{[m]^2}{m}, \quad [c_0, Q_c] = 1.$$

Define the generating functions for the Drinfeld basis by $X_i^\pm(z) = \sum_{m \in \mathbf{Z}} X_m^{\pm,i} z^{-m-1}$, and introduce h_0^i by setting $K^i = q^{h_0^i}$. Define $Q_{h^1} = Q_{a^1} - Q_{a^2}$, $Q_{h^2} = Q_{a^2} + Q_b$ and $h_i(z; \beta)$ by

$$h_i(z; \beta) = - \sum_{n \neq 0} \frac{h_n^i}{[n]} q^{-\beta|n|} z^{-n} + Q_{h^i} + h_0^i \ln z, \quad (\text{A.1})$$

where β is a parameter. Other bosonic fields are defined similarly.

The Drinfeld generators at level-one are realized by the free boson fields as [25]

$$h_m^1 = a_m^1 q^{-|m|/2} - a_m^2 q^{|m|/2}, \quad h_m^2 = a_m^2 q^{-|m|/2} + b_m q^{|m|/2}, \quad m \in \mathbf{Z},$$

$$X_1^\pm(z) = \pm : e^{\pm h_1(z; \pm \frac{1}{2})} : e^{\pm i\pi a_0^1}, \quad X_2^\pm(z) = : e^{h_2(z; \frac{1}{2})} e^{c(z;0)} : e^{-i\pi a_0^1},$$

$$X_2^-(z) = : e^{-h_2(z; -\frac{1}{2})} [\partial_z e^{-c(z;0)}] : e^{i\pi a_0^1}, \quad \gamma = q,$$

where $\partial_z f(z) = \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z} : O :$ stands for the usual normal ordering of O .

Consider the bosonic Fock spaces $F_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}$, generated by a_{-m}^i, b_{-m}, c_{-m} ($m > 0$) over the vacuum vectors $|\lambda_1, \lambda_2, \lambda_3, \lambda_4\rangle$,

$$F_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} = \mathbf{C}[a_{-1}^i, a_{-2}^i, \dots; b_{-1}, \dots; c_{-1}, \dots] |\lambda_1, \lambda_2, \lambda_3, \lambda_4\rangle, \quad (\text{A.2})$$

where

$$a_m^i |\lambda_1, \lambda_2, \lambda_3, \lambda_4\rangle = 0, \quad b_m |\lambda_1, \lambda_2, \lambda_3, \lambda_4\rangle = 0, \quad c_m |\lambda_1, \lambda_2, \lambda_3, \lambda_4\rangle = 0, \quad \text{for } m > 0,$$

$$|\lambda_1, \lambda_2, \lambda_3, \lambda_4\rangle = e^{\lambda_1 Q_{a^1} + \lambda_2 Q_{a^2} + \lambda_3 Q_b + \lambda_4 Q_c} |0, 0, 0, 0\rangle.$$

Introduce the following spaces

$$F_{(\alpha; \beta)} = \bigoplus_{i, j \in \mathbf{Z}} F_{\beta+i, \beta-i+j, \beta-\alpha+j, -\alpha+j}. \quad (\text{A.3})$$

It can be shown that the bosonized action of $U_q[\widehat{sl(2|1)}]$ on $F_{(\alpha; \beta)}$ is closed. To obtain the irreducible subspaces in $F_{(\alpha; \beta)}$, it is convenient to introduce a pair of fermionic currents [26, 25]

$$\eta(z) = \sum_{n \in \mathbf{Z}} \eta_n z^{-n-1} =: e^{c(z;0)} :, \quad \xi(z) = \sum_{n \in \mathbf{Z}} \xi_n z^{-n} =: e^{-c(z;0)} :,$$

The mode expansion of $\eta(z), \xi(z)$ is well defined on $F_{(\alpha; \beta)}$ for $\alpha \in \mathbf{Z}$, and it satisfies the following relation

$$\xi_m \xi_n + \xi_n \xi_m = \eta_m \eta_n + \eta_n \eta_m = 0, \quad \xi_m \eta_n + \eta_n \xi_m = \delta_{m,n}.$$

Since η_0 commutes (or anticommutes) with $U_q[\widehat{sl(2|1)}]$, η_0 plays the role of screening charge and $\eta_0 \xi_0$ qualify as the projector from $F_{(\alpha; \beta)}$ to the Kernel of η_0 . Set $\lambda_\alpha = (1 - \alpha)\Lambda_0 + \alpha\Lambda_2$, $\alpha \in \mathbf{Z}$, where Λ_i ($i = 0, 1, 2$) are the fundamental weights of $U_q[\widehat{sl(2|1)}]$, and

$$\mu_\alpha = \begin{cases} \Lambda_\alpha, & \alpha = 0, 1, 2 \\ \lambda_{\alpha-1} & \text{for } \alpha > 2 \\ \lambda_\alpha & \text{for } \alpha < 0 \end{cases}. \quad (\text{A.4})$$

Define $V(\mu_\alpha) = \eta_0 \xi_0 F_{(\alpha, \beta - \alpha)}$. Following [25, 7], $V(\mu_\alpha)$ ($\alpha \in \mathbf{Z}$) are the irreducible highest weight $U_q[\widehat{sl(2|1)}]$ -modules with the highest weight μ_α .

A.2 Level-one Vertex operators

Let $V(\lambda)$ be the highest weight $U_q[\widehat{sl(2|1)}]$ -module with the highest weight λ . Consider the following intertwiners of $U_q[\widehat{sl(2|1)}]$ -modules:

$$\Phi_\lambda^{\mu V}(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z, \quad \Phi_\lambda^{\mu V^*}(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z^{*S}.$$

They are intertwiners in the sense that for any $x \in U_q[\widehat{sl(2|1)}]$,

$$\Theta(z) \cdot x = \Delta(x) \cdot \Theta(z), \quad \Theta(z) = \Phi(z), \Phi^*(z), \quad (\text{A.5})$$

the grading of these operators is $[\Theta(z)] = 0$. $\Phi(z)$ is called type I (dual) vertex operator [9]. We expand the vertex operator as

$$\Phi(z) = \sum_{j=1,2,3} \Phi(z)_j \otimes v_j, \quad \Phi^*(z) = \sum_{j=1,2,3} \Phi^*(z)_j \otimes v_j^{*S}. \quad (\text{A.6})$$

Define the operators $\phi_j(z)$, $\phi_j^*(z)$, $\psi_j(z)$ and $\psi_j^*(z)$ ($j = 1, 2, 3$) by

$$\phi_3(z) =: e^{-h_2^*(q^2 z; -\frac{1}{2}) + c(q^2 z; 0)} : e^{-i\pi a_0^2}, \quad (\text{A.7})$$

$$\phi_2(z) = -[\phi_3(z), X_0^{-,2}]_{q^{-1}}, \quad \phi_1(z) = [\phi_2(z), X_0^{-,1}]_q, \quad (\text{A.8})$$

$$\phi_1^*(z) =: e^{h_1^*(qz; -\frac{1}{2})} : e^{i\pi a_0^2}, \quad (\text{A.9})$$

$$\phi_2^*(z) = -q^{-1}[\phi_1^*(z), X_0^{-,1}]_q, \quad \phi_3^*(z) = q^{-1}[\phi_2^*(z), X_0^{-,2}]_q, \quad (\text{A.10})$$

where $h_m^{*1} = -h_m^2$, $h_m^{*2} = -h_m^1 - \frac{[2m]}{[m]}h_m^2$ and $Q_{h^{*1}} = -Q_{h^2}$, $Q_{h^{*2}} = -Q_{h^1} - 2Q_{h^2}$. Since the operator $\phi_i(z)$, $\phi_i^*(z)$ commute (or anti-commute) with η_0 , we define

$$\Phi_i(z) = \eta_0 \xi_0 \phi_i(z) \eta_0 \xi_0, \quad \Phi_i^*(z) = \eta_0 \xi_0 \phi_i^*(z) \eta_0 \xi_0. \quad (\text{A.11})$$

According [25, 7], the vertex operators $\Phi(z)$ and $\Phi^*(z)$ (A.6) given by (A.11) are the only type I vertex operators of $U_q[\widehat{sl(2|1)}]$ which intertwine the level-one irreducible highest weight $U_q[\widehat{sl(2|1)}]$ -modules $V(\mu_\alpha)$ ($\alpha \in \mathbf{Z}$)

$$\Phi(z) : V(\mu_\alpha) \longrightarrow V(\mu_{\alpha-1}) \otimes V_z, \quad \Phi^*(z) : V(\mu_\alpha) \longrightarrow V(\mu_{\alpha+1}) \otimes V_z^{*S}.$$

It is shown [7] that the above vertex operators satisfy the graded Faddeev-Zamolodchikov algebra

$$\Phi_j(z_2) \Phi_i(z_1) = \sum_{kl} R\left(\frac{z_1}{z_2}\right)_{ij}^{kl} \Phi_k(z_1) \Phi_l(z_2) (-1)^{[i][j]}, \quad (\text{A.12})$$

$$\Phi_j^*(z_2) \Phi_i^*(z_1) = \sum_{kl} R\left(\frac{z_1}{z_2}\right)_{kl}^{ij} \Phi_k^*(z_1) \Phi_l^*(z_2) (-1)^{[i][j]}, \quad (\text{A.13})$$

$$\Phi_j(z_2) \Phi_i^*(z_1) = \sum_{kl} \bar{R}\left(\frac{z_1}{z_2}\right)_{ij}^{kl} \Phi_k^*(z_1) \Phi_l(z_2) (-1)^{[k][l]}, \quad (\text{A.14})$$

where $\bar{R}(z) = R^{-1, st_1}(z)$. Moreover, the vertex operators having the following invertibility relations

$$\Phi_i(z) \Phi_j^*|_{V(\Lambda_\alpha)} = -(-1)^{[j]} \delta_{ij} id_{V(\Lambda_\alpha)}, \quad (\text{A.15})$$

$$-\sum_k (-1)^{[k]} \Phi_k^*(z) \Phi_k(z)|_{V(\Lambda_\alpha)} = id|_{V(\Lambda_\alpha)},$$

$$\Phi_i^*(z q^2) \Phi_j(z)|_{V(\Lambda_\alpha)} = \delta_{ij} q^{2\rho_i} id|_{V(\Lambda_\alpha)}, \quad (\text{A.16})$$

$$\sum_k q^{-2\rho_k} \Phi_k(z) \Phi_k^*(z q^2)|_{V(\Lambda_\alpha)} = id|_{V(\Lambda_\alpha)}.$$

B Appendix B

In this appendix, we give the normal ordering relations of fundamental bosonic fields:

$$\begin{aligned}
e^{h_1(z_1;\beta_1)}e^{h_1(z_2;\beta_2)} &= (z_1 - q^{-(\beta_1+\beta_2)+1}z_2)(z_1 - q^{-(\beta_1+\beta_2)-1}z_2) : e^{h_1(z_1;\beta_1)}e^{h_1(z_2;\beta_2)} :, \\
e^{h_1(z_1;\beta_1)}e^{h_2(z_2;\beta_2)} &= \frac{1}{z_1 - q^{-(\beta_1+\beta_2)}z_2} : e^{h_1(z_1;\beta_1)}e^{h_2(z_2;\beta_2)} :, \\
e^{h_2(z_1;\beta_1)}e^{h_1(z_2;\beta_2)} &= \frac{1}{z_1 - q^{-(\beta_1+\beta_2)}z_2} : e^{h_2(z_1;\beta_1)}e^{h_1(z_2;\beta_2)} :, \\
e^{h_2(z_1;\beta_1)}e^{h_2(z_2;\beta_2)} &=: e^{h_2(z_1;\beta_1)}e^{h_2(z_2;\beta_2)} :, \\
e^{h_i(z_1;\beta_1)}e^{h_j^*(z_2;\beta_2)} &= (z_1 - q^{-(\beta_1+\beta_2)}z_2)^{\delta_{ij}} : e^{h_i(z_1;\beta_1)}e^{h_j^*(z_2;\beta_2)} :, \\
e^{h_i^*(z_1;\beta_1)}e^{h_j(z_2;\beta_2)} &= (z_1 - q^{-(\beta_1+\beta_2)}z_2)^{\delta_{ij}} : e^{h_i^*(z_1;\beta_1)}e^{h_j(z_2;\beta_2)} :, \\
e^{h_1^*(z_1;\beta_1)}e^{h_1^*(z_2;\beta_2)} &=: e^{h_1^*(z_1;\beta_1)}e^{h_1^*(z_2;\beta_2)} :, \\
e^{h_1^*(z_1;\beta_1)}e^{h_2^*(z_2;\beta_2)} &= \frac{1}{z_1 - q^{-(\beta_1+\beta_2)}z_2} : e^{h_1^*(z_1;\beta_1)}e^{h_2^*(z_2;\beta_2)} :, \\
e^{h_2^*(z_1;\beta_1)}e^{h_1^*(z_2;\beta_2)} &= \frac{1}{z_1 - q^{-(\beta_1+\beta_2)}z_2} : e^{h_2^*(z_1;\beta_1)}e^{h_1^*(z_2;\beta_2)} :, \\
e^{h_2^*(z_1;\beta_1)}e^{h_2^*(z_2;\beta_2)} &= \frac{1}{(z_1 - q^{-(\beta_1+\beta_2)+1}z_2)(z_1 - q^{-(\beta_1+\beta_2)-1}z_2)} : e^{h_2^*(z_1;\beta_1)}e^{h_2^*(z_2;\beta_2)} :, \\
e^{c(z_1;\beta_1)}e^{c(z_2;\beta_2)} &= (z_1 - q^{-(\beta_1+\beta_2)}z_2) : e^{c(z_1;\beta_1)}e^{c(z_2;\beta_2)} : .
\end{aligned}$$

C Appendix C

We here summarize the formulas concerning coherent states of bosons which have been used in section 5.

The coherent states $|\zeta^1, \zeta^2, \zeta^3\rangle$ and $\langle \bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3|$ in the Fock space $F_{(0;\beta)}$ and its dual space $F_{(0;\beta)}^*$ are defined by

$$|\zeta^1, \zeta^2, \zeta^3\rangle = \exp \left\{ \sum_{m=1}^2 \sum_{i=1}^3 \frac{m}{[m]^2} \zeta_m^i h_{-m}^{*i} + \sum_{m=1}^3 \frac{m}{[m]^2} \zeta_m^3 c_{-m} \right\} |\beta, \beta, \beta, 0\rangle, \quad (\text{C.1})$$

$$\langle \bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3| = \langle \beta, \beta, \beta, 0| \exp \left\{ \sum_{m=1}^2 \sum_{i=1}^3 \frac{m}{[m]^2} \bar{\zeta}_m^i h_m^{*i} + \sum_{m=1}^3 \frac{m}{[m]^2} \bar{\zeta}_m^3 c_m \right\} \quad (\text{C.2})$$

where ζ_m^l and $\bar{\zeta}_m^l$ ($l = 1, 2, 3$, $m = 1, 2, \dots$) are complex conjugate parameters.

Noting that

$$\begin{aligned}
h_m^i |\beta, \beta, \beta, 0\rangle &= 0, & \langle \beta, \beta, \beta, 0| h_{-m}^i &= 0, \quad i = 1, 2, \quad m \geq 1, \\
c_m |\beta, \beta, \beta, 0\rangle &= 0, & \langle \beta, \beta, \beta, 0| c_{-m} &= 0, \quad m \geq 1,
\end{aligned}$$

one can easily verify

$$\begin{aligned}
h_m^i |\zeta^1, \zeta^2, \zeta^3\rangle &= \zeta_m^i |\zeta^1, \zeta^2, \zeta^3\rangle, & \langle \bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3| h_{-m}^i &= \bar{\zeta}_m^i \langle \bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3|, \quad i = 1, 2, \\
c_m |\zeta^1, \zeta^2, \zeta^3\rangle &= \zeta_m^3 |\zeta^1, \zeta^2, \zeta^3\rangle, & \langle \bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3| c_{-m} &= \bar{\zeta}_m^3 \langle \bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3|.
\end{aligned}$$

One can also show that the coherent states $\{|\zeta^1, \zeta^2, \zeta^3\rangle\}$ (resp. $\langle \bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3|$) form a complete basis in Fock space $F_{(0;\beta)}$ (resp. $F_{(0;\beta)}^*$). Namely, one can verify the completeness

relation

$$id_{F_{(0;\beta)}} = \int \prod_{m=1}^{\infty} \frac{d\zeta_m^1 d\bar{\zeta}_m^1 d\zeta_m^2 d\bar{\zeta}_m^2 d\zeta_m^3 d\bar{\zeta}_m^3}{\frac{[m]^2}{m} \det\left(\frac{[a_{ij}m][m]}{m}\right)} \exp \left\{ - \sum_{m=1}^{\infty} \sum_{i,j=1}^2 \frac{K_{ij}(m)m}{[m]} \zeta_m^i \bar{\zeta}_m^j \right\} \times |\zeta^1, \zeta^2, \zeta^3 \rangle \langle \bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3|, \quad (C.3)$$

where $K_{ij}(n)$ is a 2×2 matrix satisfying

$$\sum_{l=1}^2 K_{il}(n) [a_{lj}n] = \delta_{ij}.$$

One may also derive the following identity

$$\begin{aligned} & \int \prod_{m=1}^{\infty} \frac{d\zeta_m^1 d\bar{\zeta}_m^1 d\zeta_m^2 d\bar{\zeta}_m^2 d\zeta_m^3 d\bar{\zeta}_m^3}{\frac{[m]^2}{m} \det\left(\frac{[a_{ij}m][m]}{m}\right)} \exp \left\{ - \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \left(\bar{\zeta}_m^1, \bar{\zeta}_m^2, \bar{\zeta}_m^3, \zeta_m^1, \zeta_m^2, \zeta_m^3 \right) \mathcal{A}_m \begin{pmatrix} \bar{\zeta}_m^1 \\ \bar{\zeta}_m^2 \\ \bar{\zeta}_m^3 \\ \zeta_m^1 \\ \zeta_m^2 \\ \zeta_m^3 \end{pmatrix} \right. \\ & \quad \left. + \sum_{m=1}^{\infty} (\bar{\zeta}_m^1, \bar{\zeta}_m^2, \bar{\zeta}_m^3, \zeta_m^1, \zeta_m^2, \zeta_m^3) \mathcal{B}_m \right\} \\ & = \prod_{m=1}^{\infty} (-\det \mathcal{A}_m)^{-\frac{1}{2}} \exp \left\{ \frac{1}{2} \sum_{m=1}^{\infty} \frac{[m]^2}{m} \det \left(\frac{[a_{ij}m][m]}{m} \right) \mathcal{B}_m^t \mathcal{A}_m^{-1} \mathcal{B}_m \right\}, \end{aligned} \quad (C.4)$$

where \mathcal{A}_m are invertible constant 6×6 matrices and \mathcal{B}_m are constant 6 component vectors.

References

- [1] F.H.L. Essler, V.E. Korepin, K. Schoutens, *Phys. Rev. Lett.* **68** (1992), 2960.
- [2] A. Foerster, M. Karowski, *Nucl. Phys.* **B396** (1993), 611.
- [3] A.J. Bracken, M.D. Gould, J.R. Links, Y.-Z. Zhang, *Phys. Rev. Lett* **74** (1995), 2768.
- [4] M.P. Pfannmuller, H. Frahm, *Nucl. Phys.* **B479** (1996), 575.
- [5] P.B. Ramos, M.J. Martins, *Nucl. Phys.* **B479** (1996), 678.
- [6] H. Fan, M. Wadati, X.M. Wang, *Phys. Rev.* **B61**(2000), 3450.
- [7] W.-L. Yang, Y.-Z. Zhang, *Nucl. Phys.* **B547** (1999), 599; *Jour. Math. Phys.* **41** (2000), 5849.
- [8] B. Davies, O. Foda, M. Jimbo, T. Miwa, A. Nakayashiki, *Commun. Math. Phys.* **151**(1993) , 89.
- [9] M. Jimbo, T. Miwa, *Algebraic analysis of solvable lattice model*, CBMS Regional Conference Series in Mathematics, **Vol. 85** (AMS, Providence, 1994).
- [10] I.B. Frenkel, N.Yu. Reshetikhin, *Commun. Math. Phys.* **146** (1992), 1.

- [11] M.Idzumi, *Int. J. Mod. Phys.* **A9** (1994), 449; H.Bougurzi and R.Weston , *Nucl. Phys.* B417 (1994), 439.
- [12] Y.Koyama, *Commun. Math. Phys.* **164**(1994), 277.
- [13] S.Lukyanov and Y.Pugai, *Nucl. Phys.* **B473**(1996), 631; Y.Asai, M.Jimbo, T.Miwa and Y.Pugai, *Jour. Phys.* **A29** (1996), 6595.
- [14] J.Hong, S.J.Kang, T.Miwa and R.Weston, *J. Phys.* **A31** (1998), L515.
- [15] B.Y. Hou, W.-L. Yang, Y.-Z. Zhang, *Nucl. Phys.* **B556** (1999), 485.
- [16] M. Jimbo, R. Kedem, T. Kojima, H. Konno, T. Miwa, *Nucl. Phys.* **B441** (1995), 437.
- [17] E.K. Sklyanin, *Jour. Phys.* **A21** (1988), 2375.
- [18] S. Ghoshal, A. Zamolodchikov, *Int. J. Mod. Phys. A* **21**(1994), 3841.
- [19] T. Miwa, R. Weston, *Nucl. Phys.* **B** (1997), 517; B.Y. Hou, W.-L. Yang, *Commun. Theor. Phys.* **27** (1997), 257.
- [20] H. Furutsu, T. Kojima, e-print *solv-int/9905009*; T. Kojima, Y. -H. Quano, e-print *nlin.SI/0001038*.
- [21] W.-L. Yang, Y.-Z. Zhang, Izergin-Korepin model with a boundary, *Nucl. Phys.* **B** (2001), in press.
- [22] B. Y. Hou, K.J. Shi, Y.S. Wang, W.-L. Yang, *Jour. Phys.* **A30**(1997), 251; *Int. Jour. Mod. Phys.* **A12** (1997), 1711; H. Furutsu, T. Kojima, Y.-H. Quano, e-print *solv-int/9910012*.
- [23] B.Y. Hou, K. J. Shi, W.-L. Yang, *Commun. Theor. Phys.* **31** (1999), 265; A.Doikou, R. Nepomechie, *Phys. Lett.* **B462** (1999), 121.
- [24] A.J. Bracken, X.Y. Ge, Y.-Z. Zhang, H.Q. Zhou, *Nucl. Phys.* **B516** (1998), 588.
- [25] K. Kimura, J. Shiraishi, J. Uchiyama, *Commun. Math. Phys.* **188** (1997), 367.
- [26] P. Bouwknegt, A. Ceresole, J. G. McCarthy, P. van Nieuwenhuizen, *Phys. Rev.* **D39** (1989), 2971.